

# Thermodynamics of a Weakly Interacting Bose-Einstein Gas

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**Abstract:** The one-loop effective potential for non-relativistic bosons with a delta function repulsive potential is calculated for a given chemical potential using functional methods. After renormalization and at zero temperature it reproduces the standard ground state energy and pressure as function of the particle density. At finite temperatures it is found necessary to include ring corrections to the one-loop result in order to satisfy the Goldstone theorem. It is natural to introduce an effective chemical potential directly related to the order parameter and which uniformly decreases with increasing temperatures. This is in contrast to the ordinary chemical potential which peaks at the critical temperature. The resulting thermodynamics in the condensed phase at very low temperatures is found to be the same as in the Bogoliubov approximation where the degrees of freedom are given by the Goldstone bosons. At higher temperatures the ring corrections dominate and result in a critical temperature unaffected by the interaction.

## 1 Introduction

Bose-Einstein condensation is central to much of our understanding of phenomena in condensed matter physics [1]. It is one of the simplest processes where quantum effects manifest themselves on the macroscopic level when a finite fraction of the non-interacting bosons in a system start to occupy the lowest energy level. Although all the particles will then be in the same quantum state at zero temperature, this condensate does not have real long-range order and does not truly represent a different phase. It was Bogoliubov [2] who first showed that a short-range repulsion between the particles is necessary in order to have a real condensate with the particles in a new physical phase which is superfluid. His description of the condensation of interacting bosons has since then formed the basis for a much more detailed understanding of these important phenomena [3].

Until very recently the only physical Bose-Einstein system of non-relativistic particles exhibiting a phase transition at low temperatures, was liquid  $He^4$ . But here the particle density is so high that it is a strongly interacting system so that

perturbation theory around the free system does not work [3]. However, with the recent experimental progress made in connection with magnetically trapped bosons in the gas phase [4], the situation has radically changed and systems of weakly interacting bosons can now be studied. These were theoretically investigated in a series of papers by Lee, Yang and their collaborators forty years ago using methods from statistical mechanics [5, 6, 7, 8]. Their many results still represent to a large degree the state of theoretical understanding of weakly interacting boson gases.

In the normal phase one can assign a definite number of particles to the system while this is impossible after the condensate has formed [9]. Bose-Einstein condensation of interacting particles is therefore the oldest and still probably the simplest example of spontaneous symmetry breakdown which today lies at the very heart of modern elementary particle theory [10]. Since these theories are relativistic, it is not obvious how Bogoliubov's method can be used in this case. Instead one has developed very powerful methods based on Feynman's path integral formulation of quantum field theories [11] which allow the calculation of the corresponding effective potentials in a very systematic way [12, 13, 14].

This approach to spontaneous symmetry breakdown based upon functional methods has not yet been used to the same extent in the study of Bose-Einstein condensation of non-relativistic systems although the basic elements are already in a modern textbook [15]. In two dimensions it has been used by Lozano[16]. It is not clear if this approach is equivalent to the Bogoliubov method or not. This is what we have set out to investigate here. A first step in this direction was taken several years ago by Kapusta [17] who considered interacting systems of relativistic bosons at non-zero chemical potential and their condensation at low temperatures. More recently, Bernstein and Dodelson [18] and Benson, Bernstein and Dodelson [19] extended these relativistic calculations and also considered the non-relativistic limit. We will here show that their finite-temperature results are incomplete in that they have not included the contributions from the ring or daisy diagrams which are known to be essential at non-zero temperatures [13, 14, 20, 21]. Since then, the non-relativistic Bose gas has also been studied by Stoof and Bijlsma [22].

The use of functional methods in quantum statistical physics of non-relativistic systems has not yet as widespread as for relativistic systems. One of the best introductions have been given by Popov [23]. Our approach is different and more along the lines used in relativistic quantum field theories [24], but we will to a large degree reproduce his results. The necessary formalism is established in the next section where we will derive the thermodynamics of a gas of free bosons in this language. We will work with the two real components of the field instead of the complex field itself and its conjugate which is usually done in condensed matter physics [25, 26]. We find this choice of variables especially advantageous in the case

of interacting particles at zero temperature considered in Section 3. We calculate the effective potential and free energy at non-zero chemical potential in the one-loop approximation where we include the quantum effects of the fluctuations around the classical solution. After removing the divergences in the theory by renormalization of the coupling constant and the chemical potential, we find the ground state energy of the hard-core bosons to be in exact agreement with the standard results of Lee and Yang [5]. In Section 4 we extend the calculation of the effective potential to finite temperatures using the imaginary-time formalism. We discover that the one-loop approximation is no longer consistent with the Goldstone theorem requiring the excitation spectrum to be linear in the long wavelength limit. The problem is solved by including the so-called daisy or ring corrections to the boson propagator. These important contributions to the free energy of relativistic quantum field theories at finite temperature were first discussed by Dolan and Jackiw [13] and Weinberg [14]. They have recently become of importance in the connection with the standard model of elementary particles at finite temperatures [20, 21]. In the last section we discuss the obtained results and compare with what has been obtained by other methods. This functional approach allows now in principle a systematic calculation of higher loop corrections to the thermodynamics of the interacting boson gas.

## 2 Functional methods for the non-relativistic boson gas

The wavefunction  $\psi = \psi(\mathbf{x}, t)$  for a free, non-relativistic particle of mass  $m$  satisfies the Schrödinger equation

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi \quad (1)$$

when we use units so that  $\hbar = 1$ . In the second quantized description of a system of many such particles  $\psi(\mathbf{x}, t)$  becomes the corresponding quantum field. The wave equation (1) is then the classical equation of motion. It follows from the Schrödinger Lagrangian

$$\mathcal{L} = i\psi^*\partial_t\psi - \frac{1}{2m}|\nabla\psi|^2 \quad (2)$$

Constructing now the Hamiltonian  $H$  and the number operator  $N = \int d^3x \psi^*\psi$ , the grand canonical partition function for the is then given by

$$\Xi(\beta, \mu) = \text{Tr} e^{-\beta(H - \mu N)} \quad (3)$$

where  $\beta = 1/T$  when the system is in thermal equilibrium at temperature  $T$  and chemical potential  $\mu$ . The Boltzmann constant is taken to be  $k_B = 1$ . Rewriting now the trace as a path integral [11], one obtains the functional integral

$$\Xi(\beta, \mu) = \int \mathcal{D}\psi \mathcal{D}\psi^* e^{-\int_0^\beta d\tau \int d^3x \mathcal{L}_E(\psi^*, \psi)} \quad (4)$$

where the field  $\psi = \psi(\mathbf{x}, \tau)$  is now function of imaginary time  $\tau = it$ . Its dynamics is governed by the Euclidean Lagrangian density

$$\mathcal{L}_E = \psi^* \partial_\tau \psi + \frac{1}{2m} |\nabla \psi|^2 - \mu \psi^* \psi \quad (5)$$

where we have included the contribution from the chemical potential.

## 2.1 Complex field formalism

Since the integral (4) is Gaussian, it can be immediately evaluated. The result for the free energy  $\Omega = -\ln \Xi/\beta$  can then be written as

$$\Omega(\beta, \mu) = \frac{1}{\beta} \text{Tr} \ln(\partial_\tau - \nabla^2/2m - \mu) \quad (6)$$

The functional trace is given by eigenvalues of the indicated operator. They are found by expanding the Bose field in plane waves

$$\psi(\mathbf{x}, \tau) = \sqrt{\frac{1}{\beta V}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{k}} \psi_{n\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x} + i\omega_n \tau} \quad (7)$$

where  $\omega_n = 2\pi n/\beta$  are the corresponding Matsubara frequencies. Then we have

$$\Omega(\beta, \mu) = \frac{1}{\beta} \sum_{\mathbf{k}} \sum_{n=-\infty}^{\infty} \ln(-i\omega_n + \varepsilon_{\mathbf{k}} - \mu) \quad (8)$$

where  $\varepsilon_{\mathbf{k}} = \mathbf{k}^2/2m$  is the single-particle energy. One can regularize the divergent sum by taking the derivative with respect to  $\varepsilon_{\mathbf{k}} = \mathbf{k}^2/2m$ . Using then the standard sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \omega^2} = \frac{\beta}{\omega} \left[ \frac{1}{2} + \frac{1}{e^{\beta\omega} - 1} \right] \quad (9)$$

and integrating back, we have

$$\Omega(\beta, \mu) = \sum_{\mathbf{k}} \left[ \frac{1}{2} \varepsilon_{\mathbf{k}} + T \ln \left( 1 - e^{-\beta \varepsilon_{\mathbf{k}}} \right) \right] \quad (10)$$

after discarding an infinite constant. The pressure in the gas is now simply  $P = -\Omega/V$ . Taking the infinite-volume limit, it becomes

$$P(\beta, \mu) = - \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2}(\varepsilon_{\mathbf{k}} - \mu) + T \ln \left( 1 - e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)} \right) \right] \quad (11)$$

The first term is the zero-point energy which is the only contribution at zero temperature and is usually without physical content. However, in the next section where we discuss the interacting theory, it will be important.

In order to find the equation of state, we want the pressure as a function of the density  $\rho = \partial P / \partial \mu$ . Ignoring the zero-point energy in (11), we get the ordinary result

$$\rho = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} - 1} \quad (12)$$

from which one can calculate the chemical potential  $\mu$  as function of temperature and density. From the pressure one then has the equation of state. It involves a critical line, determined by  $\mu = 0$ , i.e.

$$\rho = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\varepsilon_{\mathbf{k}}} - 1} = \zeta(3/2) \left( \frac{mT}{2\pi} \right)^{3/2} \quad (13)$$

For densities above the critical value, the pressure in the gas is seen to be independent of the density. This condensed phase has thus an infinite compressibility which shows that it is non-physical. A short-ranged, repulsive potential between the particles will solve this problem and the Bose-Einstein condensate will become a physical superfluid [1].

## 2.2 Real field formalism

When we later consider the interacting gas, we will find it more convenient to take the two real components of the complex field  $\psi$  as independent field variables,

$$\psi = \sqrt{\frac{1}{2}}(\psi_1 + i\psi_2) \quad (14)$$

The Euclidean Lagrangian (5) can then be written as

$$\mathcal{L}_E = \frac{1}{2} \psi_a \widehat{M}_{ab} \psi_b \quad (15)$$

where equal indices are summed from 1 to 2. We have here introduced the matrix operator

$$\widehat{M}_{ab} = i\epsilon_{ab}\partial_\tau - \left(\frac{\nabla^2}{2m} + \mu\right)\delta_{ab} \quad (16)$$

where  $\epsilon_{ab}$  is the antisymmetric tensor in two dimensions with  $\epsilon_{12} = 1$ . After expressing the corresponding action in terms of the complex Fourier components in (7), the partition function (4) is now given by the functional integral

$$\Xi(\beta, \mu) = \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \exp \left[ -\frac{1}{2} \sum_{n, \mathbf{k}} (e_{1\mathbf{k}} \psi_1^* \psi_1 + e_{2\mathbf{k}} \psi_2^* \psi_2) + \sum_{n, \mathbf{k}} \omega_n (\psi_1^* \psi_2 - \psi_2^* \psi_1) \right] \quad (17)$$

The two energies  $e_{1\mathbf{k}}$  and  $e_{2\mathbf{k}}$  are here actually both equal to  $e_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ . It should also be clear that in this integral we have suppressed the Fourier indices on the field components.

From the form of the partition function (17) we see that in terms of the real field components, the free, non-relativistic field theory is at the fundamental level interacting. The diagonal part of the action moves the fields  $\psi_1$  and  $\psi_2$  at fixed time with the propagators

$$\text{---} : D_{11}^{(0)} = \frac{1}{e_{1\mathbf{k}}} \quad \text{-----} : D_{22}^{(0)} = \frac{1}{e_{2\mathbf{k}}} \quad (18)$$

while the kinetic term provides the interaction with the simple vertex

$$\text{---} \bigcirc \text{-----} : \omega_n \quad \text{-----} \bigcirc \text{---} : -\omega_n \quad (19)$$

of strength given by the Matsubara frequency. The full, free energy is now obtained in standard perturbation theory which is almost trivially solved to all orders in the interaction. First, we need the partition function of the free theory

$$\begin{aligned} \Xi_0 &= \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \exp \left[ -\frac{1}{2} \sum_{n, \mathbf{k}} (e_{1\mathbf{k}} \psi_1^* \psi_1 + e_{2\mathbf{k}} \psi_2^* \psi_2) \right] \\ &= \det^{-\frac{1}{2}}(e_{1\mathbf{k}}) \det^{-\frac{1}{2}}(e_{2\mathbf{k}}) \end{aligned} \quad (20)$$

Taking the logarithm, we find the non-interacting result

$$\begin{aligned} \beta\Omega_0 &= \frac{1}{2} [\text{Tr} \ln e_{1\mathbf{k}} + \text{Tr} \ln e_{2\mathbf{k}}] \\ &= - \left[ \frac{1}{2} \ln \bigcirc + \frac{1}{2} \ln \bigcirc \right] \end{aligned} \quad (21)$$

where the closed loops denotes the trace over the variables in the propagators. The kinetic interaction will now perturb the fields in these two loop diagrams. A 1-field will be converted two a 2-field and vice versa with the coupling constant  $\omega_n$ . Since the free propagator  $\langle \psi_1 \psi_2 \rangle_0 = 0$ , only loops with an even number of interactions will contribute, i.e. with the same number of 1- and 2-fields. We then find for the full free energy

$$\begin{aligned}
\beta\Omega &= - \left[ \frac{1}{2} \ln \bigcirc + \frac{1}{2} \ln \bigcirc\!\!\!\bigcirc + \frac{1}{2} \bigcirc\!\!\!\bigcirc + \frac{1}{4} \bigcirc\!\!\!\bigcirc\!\!\!\bigcirc + \frac{1}{6} \bigcirc\!\!\!\bigcirc\!\!\!\bigcirc\!\!\!\bigcirc + \dots \right] \\
&= \frac{1}{2} \sum_{n,\mathbf{k}} \left[ \ln(e_{1\mathbf{k}} e_{2\mathbf{k}}) + \frac{\omega_n^2}{1(e_{1\mathbf{k}} e_{2\mathbf{k}})} - \frac{\omega_n^4}{2(e_{1\mathbf{k}} e_{2\mathbf{k}})^2} + \frac{\omega_n^6}{3(e_{1\mathbf{k}} e_{2\mathbf{k}})^3} + \dots \right] \\
&= \frac{1}{2} \sum_{n,\mathbf{k}} \ln(\omega_n^2 + e_{1\mathbf{k}} e_{2\mathbf{k}})
\end{aligned} \tag{22}$$

Again regularizing as in (8) and summing over the Matsubara frequencies with the help of (9), we recover the the standard Bose-Einstein free energy (10).

## 2.3 Free propagators of real fields

The propagators (18) move only the fields at fixed time. Motion in time is induced by the kinetic interaction in (15). Its full effect can easily be calculated in perturbation theory. For the 1-field, when we again consider a Fourier mode with a given momentum  $\mathbf{k}$  and Matsubara energy  $\omega_n$ , we find:

$$\begin{aligned}
D_{11} &= \langle \psi_1 \psi_1^* \rangle = \text{—————} \\
&= \text{———} + \text{———} \bigcirc \text{-----} \bigcirc \text{———} + \text{———} \bigcirc \text{-----} \bigcirc \text{———} \bigcirc \text{-----} \bigcirc \text{———} + \dots \\
&= D_{11}^{(0)} + D_{11}^{(0)}(\omega_n) D_{22}^{(0)}(-\omega_n) D_{11}^{(0)} \\
&+ D_{11}^{(0)}(\omega_n) D_{22}^{(0)}(-\omega_n) D_{11}^{(0)}(\omega_n) D_{22}^{(0)}(-\omega_n) D_{11}^{(0)} + \dots \\
&= \frac{D_{11}^{(0)}}{1 + \omega_n^2 D_{11}^{(0)} D_{22}^{(0)}} = \frac{e_{2\mathbf{k}}}{\omega_n^2 + e_{1\mathbf{k}} e_{2\mathbf{k}}}
\end{aligned} \tag{23}$$

Similarly, we find

$$\begin{aligned}
D_{22} &= \langle \psi_2 \psi_2^* \rangle = \text{-----} \\
&= \text{-----} + \text{-----} \bigcirc \text{———} \bigcirc \text{-----} + \text{-----} \bigcirc \text{———} \bigcirc \text{-----} \bigcirc \text{———} \bigcirc \text{-----} + \dots \\
&= \frac{e_{1\mathbf{k}}}{\omega_n^2 + e_{1\mathbf{k}} e_{2\mathbf{k}}}
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
D_{12} &= \langle \psi_1 \psi_2^* \rangle = \text{---}\bigcirc\text{---} \\
&= \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---}\bigcirc\text{---}\bigcirc\text{---} + \dots \\
&= \frac{\omega_n}{\omega_n^2 + e_{1\mathbf{k}}e_{2\mathbf{k}}}
\end{aligned} \tag{25}$$

while  $D_{21} = -D_{12}$ . These results can simply be summed up in the Dyson-Schwinger equations for the full propagators

$$D_{ab} = D_{ab}^{(0)} + D_{ac}^{(0)} \Pi_{cd}^{(0)} D_{db} \tag{26}$$

With the free propagator (18),  $D_{12}^{(0)} = 0$  and the non-diagonal self energy  $\Pi_{cd}^{(0)} = \epsilon_{cd}\omega_n$ , the equations are easily solved to give the same results as above.

We will later use the Schwinger-Dyson equations to calculate the propagators when the bosons have a short-range interaction. Here the fields are essentially free and the propagators can be obtained directly from the matrix operator (16). Its Fourier transform is

$$M_{ab} = \begin{pmatrix} e_{1\mathbf{k}} & -\omega_n \\ \omega_n & e_{2\mathbf{k}} \end{pmatrix} \tag{27}$$

Taking the inverse, we then simply have

$$D_{ab} = \langle \psi_a \psi_b \rangle = M_{ab}^{-1} = \frac{1}{\omega_n^2 + e_{1\mathbf{k}}e_{2\mathbf{k}}} \begin{pmatrix} e_{2\mathbf{k}} & \omega_n \\ -\omega_n & e_{1\mathbf{k}} \end{pmatrix} \tag{28}$$

which is seen to agree with the previous results.

### 3 Hard-core bosons in the one-loop approximation

We will here consider the idealized case of bosons having only a repulsive interaction potential  $V(\mathbf{r})$  at short distances. The thermodynamics of the gas will then be mostly independent of the detailed shape of the potential which will only enter the results via the  $S$ -wave scattering length [3]

$$a = \frac{m}{4\pi} \int d^3r V(\mathbf{r}) \tag{29}$$

which is positive. This is equivalent to saying that the potential is a  $\delta$ -function, i.e.  $V(\mathbf{r}) = 2\lambda \delta(\mathbf{r})$  with the coupling constant  $\lambda = 2\pi a/m$ . In the second quantized



theory it will correspond to an interaction term  $\lambda(\psi^*\psi)^2$  in the Lagrangian. While the coupling constant  $\lambda$  would be dimensionless in the corresponding relativistic theory, it is not in the non-relativistic description we use here. Let us comment briefly upon this point.

The Lagrangian of a real relativistic scalar field  $\Psi$  can be written as

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\Psi\partial^\mu\Psi - \frac{m^2}{2}\Psi^2 - \lambda_0\Psi^4 \quad (30)$$

in real time formalism. Here  $\mu = 0 \dots 3$  and  $\lambda_0$  is the relativistic coupling constant. Since the action must be dimensionless, the field takes on the same dimension as the inverse time or inverse distance in units where the velocity of light  $c = 1$ . The coupling  $\lambda_0$  is therefore dimensionless. We now take the non-relativistic limit by letting  $m \rightarrow \infty$ . Before doing so, we introduce the non-relativistic field  $\psi$  through

$$\Psi = \frac{1}{\sqrt{2m}} \left( e^{-imt}\psi + e^{imt}\psi^* \right) \quad (31)$$

This leads to a number of terms in the Lagrangian which oscillate with frequency  $2m$ . They may be dropped as  $m \rightarrow \infty$ . The resulting non-relativistic Lagrangian takes the form

$$\mathcal{L} = i\psi^*\partial_t\psi - \frac{1}{2m}|\nabla\psi|^2 - \lambda(\psi^*\psi)^2 \quad (32)$$

with a non-relativistic coupling constant  $\lambda = 3\lambda_0/2m^2$ . This coupling is obviously not dimensionless.

### 3.1 The classical ground state

Including the above interaction, the *Euclidean* Lagrangian (5) describing the bosons is changed into

$$\mathcal{L}_E = \psi^*\partial_\tau\psi + \frac{1}{2m}|\nabla\psi|^2 - \mu\psi^*\psi + \lambda(\psi^*\psi)^2 \quad (33)$$

In the classical limit at zero temperature the system will be in the lowest energy state. The field will then attain a constant value given by the minimum of the classical potential

$$U(\psi) = -\mu\psi^*\psi + \lambda(\psi^*\psi)^2 \quad (34)$$

It is invariant under the  $U(1)$  phase transformation  $\psi(\mathbf{x}) \rightarrow e^{i\theta}\psi(\mathbf{x})$  and thus depends only on the modulus  $|\psi|$  of the field. In Fig.1 the classical potential is plotted

for the two cases  $\mu > 0$  and  $\mu < 0$ . We see that in the first case, we will have a ground state with spontaneous breakdown of the  $U(1)$  symmetry in which the field takes the classical value  $|\psi| = \sqrt{\mu/2\lambda}$ . We will in later sections see that  $\mu$  will be negative at high temperatures and the gas will be in the normal state with  $|\psi| = 0$ .

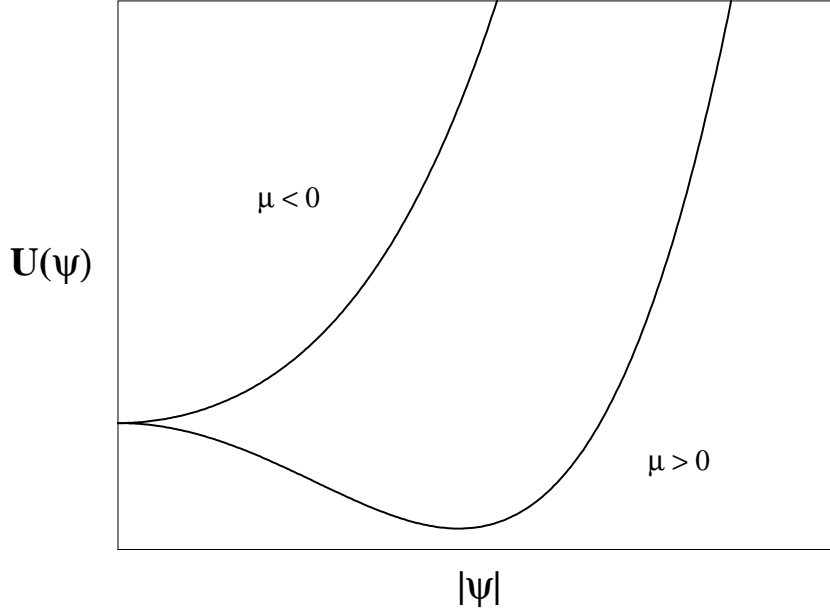


Figure 1: The classical potential  $U(\psi)$  plotted as function of the field modulus  $|\psi|$  in arbitrary units.

To be more quantitative, the symmetry-broken ground state has a classical pressure given by the minimum of the potential (34), i.e.  $P(\mu) = \mu^2/4\lambda$ . The corresponding number density is then  $\rho = \mu/2\lambda$  and therefore  $P = \lambda\rho^2$ . This is also the classical ground state energy density  $\mathcal{E}(\rho)$  as follows from the thermodynamic Legendre transform

$$\mathcal{E}(\rho) = \mu\rho - P(\mu) \quad (35)$$

In this way we have already cured one of the problems of the ideal Bose-Einstein gas, namely the infinite compressibility of the condensed phase. What is needed next is the inclusion of quantum and thermal fluctuations around this classical ground state.

### 3.2 Gaussian fluctuations

Denoting the fluctuating field by  $\chi$ , we can write the full Bose field as

$$\psi(x) = \sqrt{\frac{1}{2}}[v + \chi(x)] \quad (36)$$

where  $v$  is the constant field of the condensate. Inserting this into the Lagrangian (33), we can rewrite it as

$$\mathcal{L}_E = -\frac{\mu}{2}|v|^2 + \frac{\lambda}{4}|v|^4 + \frac{1}{2}\chi_a \widehat{M}_{ab} \chi_b + \lambda(v \cdot \chi)(\chi \cdot \chi) + \frac{\lambda}{4}(\chi \cdot \chi)^2 \quad (37)$$

when expressed in terms of the two real components of the complex fields  $\chi$  and  $v$ . Terms linear in  $\chi$  have been dropped since they will not contribute after integration over space. The matrix operator  $\widehat{M}_{ab}$  is now

$$\widehat{M}_{ab} = i\epsilon_{ab}\partial_\tau - \left( \frac{\nabla^2}{2m} - \lambda(v \cdot v) + \mu \right) \delta_{ab} + 2\lambda v_a v_b \quad (38)$$

Due to the  $U(1)$  symmetry of the system, we can choose the classical field  $v$  to be real. Taking the Fourier transform of the operator, it becomes

$$M_{ab} = \begin{pmatrix} \varepsilon_{\mathbf{k}} - \mu + 3\lambda v^2 & -\omega_n \\ \omega_n & \varepsilon_{\mathbf{k}} - \mu + \lambda v^2 \end{pmatrix} \quad (39)$$

where again  $\varepsilon_{\mathbf{k}} = \mathbf{k}^2/2m$ .

The grand canonical partition function (4) can be now written as a functional integral over the real fields  $\chi_1$  and  $\chi_2$  as

$$\Xi(\beta, \mu) = e^{-\beta V(-\frac{\mu}{2}v^2 + \frac{\lambda}{4}v^4)} \int \mathcal{D}\chi_1 \mathcal{D}\chi_2 e^{-\int_0^\beta d\tau \int d^3x [\frac{1}{2}\chi_a \widehat{M}_{ab} \chi_b + \mathcal{L}_{int}]} \quad (40)$$

where  $V$  is the volume of the system and

$$\mathcal{L}_{int} = \lambda v \chi_1 (\chi \cdot \chi) + \frac{\lambda}{4}(\chi \cdot \chi)^2 \quad (41)$$

gives the interactions of the fluctuating field. The contribution from the first part of the Lagrangian being quadratic in the fields, can be evaluated as for the free theory in Section 2 and will be the one-loop result. Higher loop corrections due to the cubic and quartic terms in  $\mathcal{L}_{int}$  can then be systematically calculated in perturbation theory. These will be especially important at finite temperatures and will be considered in the next section.

In the one-loop approximation, we keep only the quadratic part of the Lagrangian. The integral (40) is then Gaussian and gives the corresponding free energy

$$\frac{1}{V} \Omega(\mu, v) = -\frac{\mu}{2} v^2 + \frac{\lambda}{4} v^4 + \frac{1}{2\beta V} \ln \det M \quad (42)$$

From (39) we find the determinant  $\det M = \prod_{n,\mathbf{k}} (\omega_n^2 + \omega_{\mathbf{k}}^2)$  where the excitation energy is now

$$\omega_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} - \mu + \lambda v^2)(\varepsilon_{\mathbf{k}} - \mu + 3\lambda v^2)} \quad (43)$$

Summing over the Matsubara frequencies using (9) again, we obtain for the free energy (42)

$$\frac{1}{V} \Omega(\mu, v) = -\frac{\mu}{2} v^2 + \frac{\lambda}{4} v^4 + \frac{1}{V} \sum_{\mathbf{k}} \left[ \frac{1}{2} \omega_{\mathbf{k}} + T \ln (1 - e^{-\beta \omega_{\mathbf{k}}}) \right] \quad (44)$$

As a function of the condensate  $v$ , this is the effective potential  $U_{eff}$  for the system in the one-loop approximation. The thermodynamical free energy is the value of the function in its minimum and equals the negative of the pressure  $P(\mu, T)$ . Due to the quantum fluctuations, this will be slightly shifted away from the classical minimum at  $v^2 = \mu/\lambda$ . For this value of the condensate, the dispersion relation (43) simplifies to the Bogoliubov result [2]

$$\omega = \sqrt{\varepsilon(\varepsilon + 2\mu)} = \frac{k}{2m} \sqrt{k^2 + 4m\mu} \quad (45)$$

Here, and in the following we drop the wave number index on  $\omega$ . In the long-wavelength limit  $k \rightarrow 0$  it becomes linear and represents the phonon excitations. This is a consequence of the Goldstone theorem which requires excitations with a linear dispersion relation whenever a continuous symmetry is spontaneously broken as here [9]. We will find in Section 4 that the quantum corrections will not change this dispersion relation except that the classical chemical potential  $\mu$  is replaced by an effective potential  $\bar{\mu}$  which results from the quantum self energies in the field propagators.

### 3.3 Renormalization

The infinite sum (44) is seen to be strongly divergent in the ultraviolet limit  $k \rightarrow \infty$ . A similar divergence was also found in the free theory in Section 2. We can thus remove the strongest divergence by subtracting the first term in (10) so that we

recover the pressure in the non-interacting gas. Taking the infinite-volume limit we then have for the effective potential

$$U_{eff}(v, \mu, T) = -\frac{\mu}{2}v^2 + \frac{\lambda}{4}v^4 + \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2}(\omega - \varepsilon + \mu) + T \ln(1 - e^{-\beta\omega}) \right] \quad (46)$$

However, the integral representing the Gaussian fluctuations at zero temperature is still not finite. The divergences can be removed by renormalizing the coupling constant  $\lambda$  and the chemical potential  $\mu$ . For this purpose we introduce the counter-terms

$$\begin{aligned} \mathcal{L}_{ct} &= -\delta\mu\psi^*\psi + \delta\lambda(\psi^*\psi)^2 = -\frac{1}{2}\delta\mu v^2 + \frac{1}{4}\delta\lambda v^4 + \frac{1}{4}\delta\lambda(\chi \cdot \chi)^2 \\ &\quad - \frac{1}{2}(\delta\mu - 3v^2\delta\lambda)\chi_1^2 - \frac{1}{2}(\delta\mu - v^2\delta\lambda)\chi_2^2 \end{aligned} \quad (47)$$

resulting directly from the classical potential (34). Again we have ignored terms linear in the fluctuating field  $\chi$ . At zero temperature we then get for the renormalized effective potential

$$\begin{aligned} U_{eff}(v, \mu, T=0) &= -\frac{\mu}{2}v^2 + \frac{\lambda}{4}v^4 \\ &\quad + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} (\omega - \varepsilon + \mu) - \frac{1}{2}\delta\mu v^2 + \frac{1}{4}\delta\lambda v^4 \end{aligned} \quad (48)$$

This can now be made finite by adjusting the quantities  $\delta\mu$  and  $\delta\lambda$ . The temperature-dependent contribution in (46) is finite by itself and thus the full free energy will be finite.

Explicit expressions for the counter-terms can be obtained by isolating the two leading divergences in the integral (48) by expanding the expression (43) for  $\omega$  for large values of the momentum  $k$ . Cutting off the integral at  $k = \Lambda$ , we can then rewrite  $U_{eff}$  as

$$\begin{aligned} U_{eff}(v, \mu, T=0) &= -\frac{\mu}{2}v^2 + \frac{\lambda}{4}v^4 - \frac{1}{2}\delta\mu v^2 + \frac{1}{4}\delta\lambda v^4 \\ &\quad + \frac{1}{2} \int^\Lambda \frac{d^3k}{(2\pi)^3} \left( \omega - \varepsilon + \mu - 2\lambda v^2 + \frac{\lambda^2 v^4}{2\varepsilon} \right) + \frac{1}{2} \int^\Lambda \frac{d^3k}{(2\pi)^3} \left( 2\lambda v^2 - \frac{\lambda^2 v^4}{2\varepsilon} \right) \end{aligned}$$

The divergences in the last integral are removed by taking the counter-terms to be

$$\delta\mu = \int^\Lambda \frac{d^3k}{(2\pi)^3} 2\lambda = \frac{\lambda}{3\pi^2} \Lambda^3 \quad (49)$$

and

$$\delta\lambda = \int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{\lambda^2}{\varepsilon} = \frac{\lambda^2}{\pi^2} m\Lambda \quad (50)$$

These give the same renormalized parameters as obtained by Benson, Bernstein and Dodelson [19]. Jackiw [27] has also obtained the same renormalized coupling constant from considerations of bound states of non-relativistic particles in a  $\delta$ -function potential. The unrenormalized coupling constant  $\lambda_0$  is thus given in terms of the cut-off by

$$\frac{1}{\lambda_0} = \frac{1}{\lambda} - m \frac{\Lambda}{\pi^2} \quad (51)$$

For fixed, renormalized coupling  $\lambda$  it increases with the cut-off. In the corresponding relativistic theory, this increase is logarithmic instead of linear.

Having removed the divergences, we can now let the cut-off  $\Lambda \rightarrow \infty$ . We then have the finite result

$$\begin{aligned} U_{eff}(v, \mu, T=0) &= -\frac{\mu}{2}v^2 + \frac{\lambda}{4}v^4 \\ &+ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left( \omega - \varepsilon + \mu - 2\lambda v^2 + \frac{\lambda^2 v^4}{2\varepsilon} \right) \end{aligned} \quad (52)$$

With definite values for the couplings  $\mu$  and  $\lambda$  it can be calculated by a numerical integration. As a function of the condensate value  $v$  it has the same shape as in Fig. 1. It will have an imaginary part in the region below the classical minimum at  $v^2 = \mu/\lambda$  whose physical significance has been discussed by Weinberg and Wu [28].

### 3.4 Condensate and pressure at zero temperature

Because of the quantum fluctuations, the minimum of the effective potential (52) is shifted away from the classical value. In the new minimum the derivative  $(\partial U_{eff}/\partial v)_\mu = 0$ . With

$$\left( \frac{\partial \omega}{\partial v} \right)_\mu = (2\varepsilon - 2\mu + 3\lambda v^2) \frac{1}{\omega} \quad (53)$$

we find the minimum to be at the condensate value

$$v_0^2 = \frac{\mu}{\lambda} - \int \frac{d^3k}{(2\pi)^3} \left( \frac{2\varepsilon - 2\mu + 3\lambda v^2}{\omega} - 2 + \frac{\lambda v^2}{\varepsilon} \right) \quad (54)$$

The integral represents here the effects of the quantum fluctuations. To this order in perturbation theory we can evaluate it using the classical value  $v^2 = \mu/\lambda$  for the condensate. It then gives

$$v_0^2 = \frac{\mu}{\lambda} - \int \frac{d^3k}{(2\pi)^3} \left( \frac{2\varepsilon + \mu}{\omega} + \frac{\mu}{\varepsilon} - 2 \right) \quad (55)$$

where now the Bogoliubov frequency  $\omega$  is given by (45). We will in the next section show that the quantum shift of the classical minimum is due to the self energies of the interacting fields. These will effectively change the chemical potential from the classical value  $\mu$  to a new value  $\bar{\mu}$  which is just  $\lambda$  times the right-hand side of equation (55) at zero temperature.

With the above value for the condensate in the minimum of the effective potential, we find by insertion into (52) the value for the thermodynamic pressure at zero temperature. To lowest order in the quantum correction, it can be written as

$$P(\mu) = \frac{\mu^2}{4\lambda} - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left( \omega - \varepsilon - \mu + \frac{\mu^2}{2\varepsilon} \right) \quad (56)$$

The full density of particles is now given by the derivative  $\rho = \partial P / \partial \mu$ , i.e.

$$\rho = \frac{\mu}{2\lambda} - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left( \frac{\varepsilon}{\omega} + \frac{\mu}{\varepsilon} - 1 \right) \quad (57)$$

Most of the particles are in the condensate with  $\mathbf{k} = 0$ . Their density  $\rho_c = v_0^2/2$  follows directly from (55). The density  $\rho_e = \rho - \rho_c$  of particles in excited states with  $\mathbf{k} \neq 0$  is thus

$$\rho_e = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left( \frac{\varepsilon + \mu}{\omega} - 1 \right) \quad (58)$$

It is caused by the hard-core repulsion between the particles and was also obtained by Benson, Bernstein and Dodelson [19].

With the Bogoliubov dispersion relation (45) for the excitation energy  $\omega$ , we can now evaluate the pressure in (56). Besides elementary integrations, it involves the integral

$$\int d\varepsilon \varepsilon \sqrt{\varepsilon + 2\mu} = \frac{2}{5} (\varepsilon + 2\mu)^{5/2} - \frac{4}{3} \mu (\varepsilon + 2\mu)^{3/2} \quad (59)$$

By construction, we now get only a non-zero contribution from the lower limit  $\varepsilon = 0$  of the integrals which gives

$$P(\mu) = \frac{\mu^2}{4\lambda} - \frac{8m^{3/2}}{15\pi^2} \mu^{5/2} \quad (60)$$

The last term represents the one-loop, quantum corrections to the classical result in the first term. This agrees with the original results of Lee and Yang [6] who considered the same system of interacting bosons with a hard-core repulsion within the framework of quantum statistical mechanics.

Equation (57) enables us to relate the chemical potential to the particle density. It can be obtained more directly by just taking the derivative of (60) with respect to  $\mu$ ,

$$\rho = \frac{\mu}{2\lambda} - \frac{4}{3\pi^2}(m\mu)^{3/2} \quad (61)$$

The sign of the second term here is apparently opposite to what was obtained by Bernstein and Dodelson [18]. It must be negative as here in order to have an increase in the zero-temperature pressure because of the repulsion between the particles and not. By inversion, we then obtain for the chemical potential to lowest order in perturbation theory,

$$\mu = 2\lambda\rho + \frac{8\lambda}{3\pi^2}(2m\lambda\rho)^{3/2} \quad (62)$$

Similarly, from (58) we find the density  $\rho_e = (m\mu)^{3/2}/3\pi^2$  of particles in excited states. Expressed instead in terms of the full density, it is

$$\rho_e = \frac{1}{3\pi^2}(2m\lambda\rho)^{3/2} = \frac{8\rho}{3}\sqrt{\frac{\rho a^3}{\pi}} \quad (63)$$

when the coupling constant  $\lambda$  is replaced by the scattering length  $a$ . This is just the textbook result [1].

With the chemical potential from (62), we now get for the pressure (60)

$$P(\rho) = \lambda\rho^2 + \frac{4m^{3/2}}{5\pi^2}(2\lambda\rho)^{5/2} \quad (64)$$

The corresponding energy density follows then from the Legendre transformation  $\mathcal{E} = \mu\rho - P$  and is

$$\mathcal{E}(\rho) = \lambda\rho^2 + \frac{8m^{3/2}}{15\pi^2}(2\lambda\rho)^{5/2} \quad (65)$$

It was first calculated by Lee and Yang [5] using the binary collision method and also by Lee, Huang and Yang [7] who used instead the pseudopotential method. Usually, it is expressed in terms of the scattering length and takes then the form

$$\mathcal{E} = \frac{2\pi a}{m}\rho^2 \left[ 1 + \frac{128}{15}\sqrt{\frac{\rho a^3}{\pi}} \right] \quad (66)$$

As a consistency check, we see that it reproduces the pressure (64) using the standard definition  $P = \rho^2\partial(\mathcal{E}/\rho)/\partial\rho$ .



## 4 Effective potential at finite temperature

Ignoring the counter-terms, the effective potential in the one-loop approximation and at finite temperature was obtained in (46) as function of the condensate  $v$ . The excitation energy  $\omega$  is now given by the general formula (43). Taking the derivative with respect to  $v$ , we then find that the minimum of  $U_{eff}$  is shifted to

$$v^2 = \frac{\mu}{\lambda} - \int \frac{d^3k}{(2\pi)^3} \frac{4\varepsilon - 4\mu + 6\lambda v^2}{\omega} \left[ \frac{1}{2} + \frac{1}{e^{\beta\omega} - 1} \right] \quad (67)$$

due to thermal fluctuations. Here,  $\mu$  is a function of temperature for fixed total particle density  $\rho$ . The above equation then gives the condensate as an implicit function of temperature. The contribution from thermal fluctuations is finite. Therefore, the divergences are removed by the counter-terms already introduced at zero temperature.

With thermal fluctuations present,  $v^2$  may differ considerably from  $\mu/\lambda$ . From (43) we see that the excitation energy no longer satisfies  $\omega \propto k$  in the long-wavelength limit. In other words, at finite temperature when the classical field takes on a modified value, the Goldstone theorem seems to be violated in the one-loop approximation considered up to now.

It is easy to see how the effective potential can be improved so that the Goldstone theorem is restored. At finite temperature there must be additional effects taken into consideration which changes the thermodynamic chemical potential  $\mu$  into an effective chemical potential  $\bar{\mu}$  so that the condensate is again given by  $v^2 = \bar{\mu}/\lambda$ . This will give a linear dispersion relation at low energy. The minimum of the effective potential will then be at

$$v^2 = v_0^2 - \int \frac{d^3k}{(2\pi)^3} \frac{4\varepsilon + 2\bar{\mu}}{\omega} \frac{1}{e^{\beta\omega} - 1} \quad (68)$$

where  $v_0^2$  is the zero-temperature result (55). This is now consistent with the Bogoliubov dispersion relation

$$\omega = \sqrt{\varepsilon(\varepsilon + 2\bar{\mu})} \quad (69)$$

which will be verified in the following where the ring-improved effective potential is derived.

The value of the condensate decreases with increasing temperatures and becomes zero when  $\bar{\mu} = 0$ . This defines the critical temperature for the system. At higher temperatures it is in the phase of unbroken symmetry where the dispersion relation simplifies to  $\omega = \varepsilon - \bar{\mu}$  as it is for a system of free particles.

## 4.1 Ring corrections to the effective potential

The chemical potential  $\mu$  represents the energy of adding or removing a single particle from the system of interacting bosons. It represents the self energy of the complex field  $\psi$ . We see from (39) that in the symmetry-broken phase each of the two real modes of the field has self energies equal to respectively  $\mu - 3\lambda v^2$  and  $\mu - \lambda v^2$  within the Gaussian approximation. Including the interactions in the full Lagrangian (37) these values will be modified by radiative loop corrections. When the system is at a non-zero temperature, the additional contributions to the self energies will depend upon temperature. This will then enable us to define a new, temperature-dependent chemical potential  $\bar{\mu}$ .

In the Gaussian approximation used in the previous section we found that the one-loop contribution to the effective potential (42) followed directly from the inverse of the field propagator  $D_{ab} = \langle \chi_a \chi_b \rangle$ . After Fourier transformation we know from (39) that it has the matrix form

$$D_{ab}^{-1} = \begin{pmatrix} e_{1\mathbf{k}} & -\omega_n \\ \omega_n & e_{2\mathbf{k}} \end{pmatrix} \quad (70)$$

where  $e_{1\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu + 3\lambda v^2$  and  $e_{2\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu + \lambda v^2$ . This lowest correction to the classical result is thus

$$\Omega_1 = \frac{1}{2\beta} \text{Tr} \ln D^{-1} = \frac{1}{2\beta} \sum_{n,\mathbf{k}} \ln(\omega_n^2 + e_{1\mathbf{k}} e_{2\mathbf{k}}) \quad (71)$$

which gave the standard expression (44).

The matrix (70) is the inverse of the free propagator which is used in the simplest version of the one-loop approximation. It was pointed out a long time ago in connection with the effective potential for scalar, relativistic theories [13, 14] that self energy corrections to the field propagator gave important contributions to the free energy at finite temperatures. More recently these daisy corrections have been investigated in more detail [20] and are now often sometimes called ring corrections to the effective potential. We will here see that they also play a crucial role in the non-relativistic theory in saving the Goldstone theorem at non-zero temperatures. A technique which incorporates the use of resummed propagators in a self-consistent way is the effective action for composite operators [29]. In the Appendix we show that this method leads to the same result as found here, to the order considered.

Denoting the interacting propagator by  $\bar{D}_{ab}$ , it will in general satisfy the Dyson-Schwinger equation (26). If  $\Pi_{ab}$  is the full, one-particle irreducible self energy, the equation then becomes

$$\bar{D}_{ab} = D_{ab} + D_{ac} \Pi_{cd} \bar{D}_{db} \quad (72)$$



Figure 2: A graphical representation of the Schwinger-Dyson equation (72). The thin and thick lines represent the free and interacting propagators, respectively.

as shown in Fig. 2. Using now this propagator in the one-loop approximation, we get the modified contribution

$$\bar{\Omega}_1 = \frac{1}{2\beta} \text{Tr} \ln \bar{D}^{-1} \quad (73)$$

to the effective potential. Since we can write (72) in the form

$$\bar{D}_{ab}^{-1} = D_{ab}^{-1} - \Pi_{ab} \quad (74)$$

we have

$$\begin{aligned} \bar{\Omega}_1 &= \frac{1}{2\beta} \text{Tr} \ln D^{-1} (1 - D\Pi) \\ &= \Omega_1 - \frac{1}{2\beta} \text{Tr} \left[ D\Pi + \frac{1}{2} (D\Pi)^2 + \frac{1}{3} (D\Pi)^3 + \dots \right] \end{aligned} \quad (75)$$

These additional contributions to the lowest order result are called ring corrections as seen in Fig. 3.

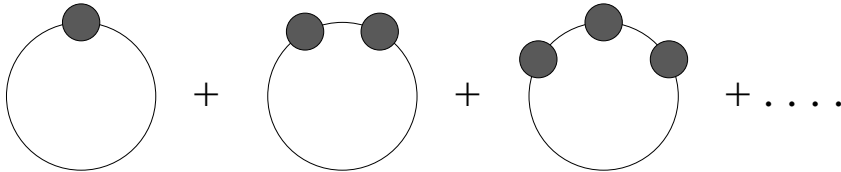


Figure 3: A sketch of the lowest order ring diagrams.

Perturbative calculations will in the following show that the off-diagonal self energies  $\Pi_{12} = -\Pi_{21}$  vanish for zero external energy. This follows in general from time-reversal invariance, which implies that the off-diagonal self energies must be odd in the Matsubara frequency  $\omega_n$ . The structure of the interacting propagator

will thus be the same as for the free propagator (70) but with the diagonal elements changed to

$$\bar{e}_{1\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu + 3\lambda v^2 - \Pi_{11} \quad (76)$$

and

$$\bar{e}_{2\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu + \lambda v^2 - \Pi_{22} \quad (77)$$

The ring-corrected one-loop contribution to the effective potential (73) is thus

$$\bar{\Omega}_1 = \frac{1}{2\beta} \sum_{n,\mathbf{k}} \ln(\omega_n^2 + \bar{e}_{1\mathbf{k}}\bar{e}_{2\mathbf{k}}) \quad (78)$$

and will be of exactly the same form as the lowest order result (44) except for the modified dispersion relation

$$\bar{\omega}_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} - \mu + 3\lambda v^2 - \Pi_{11})(\varepsilon_{\mathbf{k}} - \mu + \lambda v^2 - \Pi_{22})} \quad (79)$$

for the elementary excitations above the ground state.

The Goldstone theorem can now be satisfied also at non-zero temperatures if the chemical potential is related to the value of the field  $v$  in the minimum of the effective potential by

$$\mu = \lambda v^2 - \Pi_{22} \quad (80)$$

This is our form of the Pines-Hugenholtz relation [30] which is usually written in the complex field basis. A simple derivation of this theorem is given in [22]. Introducing the effective chemical potential

$$\bar{\mu} = \mu + \Pi_{22} \quad (81)$$

the expectation value of the field when the system is in thermal equilibrium is then given by  $v = (\bar{\mu}/\lambda)^{1/2}$  as it is at zero temperature. This will be demonstrated in the next section.

The dispersion relation (79) now becomes

$$\bar{\omega}_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}(\varepsilon_{\mathbf{k}} + 2\bar{\mu} + \Pi_{22} - \Pi_{11})} \quad (82)$$

and is by construction linear in the momentum  $k$  in the long-wavelength limit. When the self energies are calculated in the following to the lowest order in perturbation theory, the difference  $\Pi_{22} - \Pi_{11}$  will be negligible. The dispersion relation at finite

temperatures is thus the same as at zero temperature (45) when the chemical potential  $\mu$  is replaced by  $\bar{\mu}$ , i.e. it has the desired form (69). Similarly, the inverse full propagator (74) has the form

$$\bar{D}_{ab}^{-1} = \begin{pmatrix} \varepsilon_{\mathbf{k}} + 2\bar{\mu} & -\omega_n \\ \omega_n & \varepsilon_{\mathbf{k}} \end{pmatrix} \quad (83)$$

if the Goldstone theorem is to be satisfied.

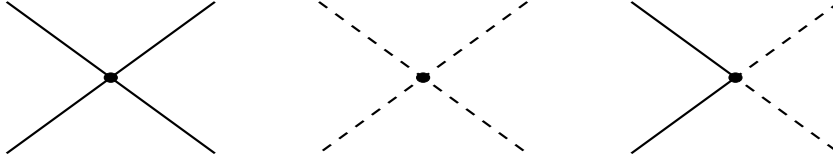


Figure 4: Four-point vertices. Solid lines represent the  $D_{11}$ -propagator and dashed lines the  $D_{22}$ -propagator.

## 4.2 One-loop contributions to the self energies

The interacting propagators and the corresponding self energies can be obtained from the full partition function  $\Xi$  of the system. It is given by the functional integral (40). In the exponent we see that there is the usual coupling  $(-\lambda)$  between four excitations  $\chi$  as shown in Fig.4 but also a new coupling of magnitude  $(-\lambda v)$  between three excitations due to the presence of the condensate  $\langle \psi \rangle = v/\sqrt{2}$ . Since we have chosen this expectation value to be real, this latter coupling will generate only the two vertices shown in Fig.5.

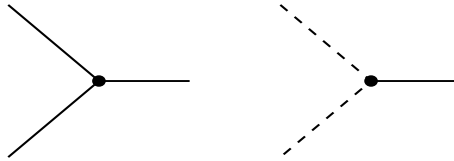


Figure 5: Three-point vertices due to the non-zero condensate.

Separating out the free partition function  $\Xi_0$ , we will here calculate the part  $\Xi_1 = \Xi/\Xi_0$  due to the interactions in lowest order of perturbation theory, i.e. include terms to order  $\lambda$ . Since the chemical potential is now defined to be  $\bar{\mu} = \lambda v^2$ , we

must include all diagrams where the four-excitation coupling occurs once or the three-excitation coupling occurs twice. Only connected diagrams will contribute to  $\ln \Xi_1$  and they are given with the corresponding combinatorial factors in Figure 6. The lines in the diagrams represent here the free propagator  $D_{ab}$  given in (70).

$$\begin{aligned}
\ln \Xi_1 = & 3/4 \text{ (two solid circles joined at a central dot)} + 3/4 \text{ (two dashed circles joined at a central dot)} + 1/2 \text{ (one solid circle joined to one dashed circle at a central dot)} \\
& + 3 \text{ (a solid circle with a horizontal line through its center and dots at both ends)} + \text{ (a dashed circle with a horizontal line through its center and dots at both ends)} + 2 \text{ (a dashed circle with a horizontal dashed line through its center and dots at both ends)} + 6 \text{ (a solid circle with a horizontal dashed line through its center and dots at both ends)} \\
& + 9/4 \text{ (three solid circles in a chain, joined at central dots)} + 3/2 \text{ (two solid circles joined at a central dot, which is then joined to a dashed circle)} + 1/4 \text{ (one solid circle joined to one dashed circle, which is then joined to another solid circle)} \\
& + 9/4 \text{ (three dashed circles in a chain, joined at central dots)} + 3/2 \text{ (two dashed circles joined at a central dot, which is then joined to a solid circle)} + 1/4 \text{ (one dashed circle joined to one solid circle, which is then joined to another dashed circle)} \\
& + 3/2 \text{ (two solid circles joined at a central dot, with a dashed line connecting the top and bottom of the central dot)} + 1/4 \text{ (one solid circle joined to one dashed circle, with a dashed line connecting the top and bottom of the central dot)} + 3/4 \text{ (two dashed circles joined at a central dot, with a dashed line connecting the top and bottom of the central dot)} \\
& + 3/4 \text{ (a solid circle with two horizontal lines through its center and dots at both ends)} + 3/4 \text{ (a dashed circle with two horizontal lines through its center and dots at both ends)} + 1/2 \text{ (a solid circle with two horizontal dashed lines through its center and dots at both ends)} + 3/2 \text{ (a dashed circle with two horizontal dashed lines through its center and dots at both ends)} \\
& + 3 \text{ (a solid circle with two horizontal dashed lines through its center and dots at both ends)} + 1/2 \text{ (a dashed circle with two horizontal dashed lines through its center and dots at both ends)} + 3 \text{ (a dashed circle with two horizontal dashed lines through its center and dots at both ends)}
\end{aligned}$$

Figure 6: Two- and three-loop diagrams contributing to  $\ln \Xi_I$ . Only two-loop diagrams are considered here. The lines represent the free propagator  $D_{ab}$ .

We can now derive the full propagator  $\bar{D}_{ab} = \langle \chi_a \chi_b \rangle$  from the partition function  $\Xi$  by taking the derivative with respect to the free propagator  $D_{ab}$  as one can see from the functional integral (40). One then finds

$$\bar{D}_{ab} = -2 \frac{\delta \ln \Xi}{\delta D_{ab}^{-1}} = -2 \frac{\delta \ln \Xi_0}{\delta D_{ab}^{-1}} + 2 D_{ca} D_{bd} \frac{\delta \ln \Xi_1}{\delta D_{cd}} \quad (84)$$

The first term is just the free propagator while the last term gives the self energy. In terms of diagrams, the derivative is obtained directly from the bubble diagrams

in Fig. 6 by opening up the corresponding lines in all of the loops. For example,  $\bar{D}_{11}$  is given by the diagrams shown in Fig. 7.

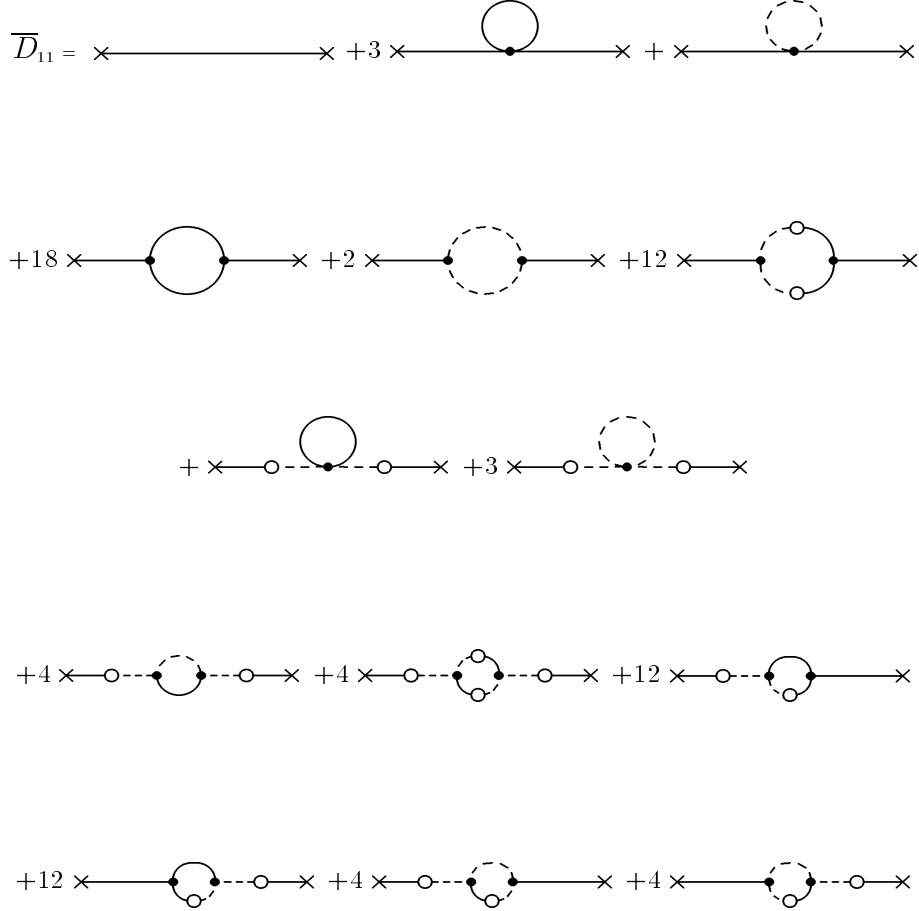


Figure 7: The lowest order diagrams contributing to the interacting propagator  $\bar{D}_{11}$ . Crosses indicate that the external propagators should be included. The lines represent the free propagator  $D_{ab}$ .

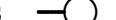

By cutting off the external lines in these diagrams, we can read off the different self energies  $\Pi_{cd}$  shown in Fig. 8. This is equivalent to using (72) in lowest order of perturbation theory where it gives  $\bar{D}_{ab} = D_{ab} + D_{ac}\Pi_{cd}D_{db}$  from which  $\Pi_{cd}$  can be isolated.

An infinite set of higher loop contributions to the free energy can now be obtained by replacing the free propagator  $D_{ab}$  in the diagrams in Fig. 6 with the full propagator  $\bar{D}_{ab}$ . This iteration then generates diagrams which correspond to the “super daisies” of Dolan and Jackiw [13]. They have more recently been investigated

$$\begin{aligned}
\Pi_{11} &= 3 \text{ (solid line with a solid loop)} + \text{ (solid line with a dashed loop)} + 18 \text{ (solid line with two solid vertices)} \\
&+ 2 \text{ (dashed line with two solid vertices)} + 12 \text{ (solid line with two solid vertices and a dashed loop)} \\
\Pi_{22} &= 3 \text{ (dashed line with a solid loop)} + \text{ (dashed line with a dashed loop)} + 4 \text{ (dashed line with two solid vertices)} + 4 \text{ (dashed line with two solid vertices and a dashed loop)} \\
\Pi_{12} &= 4 \text{ (solid line with two solid vertices and a dashed loop)} + 12 \text{ (solid line with two solid vertices and a solid loop)} \\
\Pi_{21} &= 4 \text{ (dashed line with two solid vertices and a dashed loop)} + 12 \text{ (dashed line with two solid vertices and a solid loop)}
\end{aligned}$$

Figure 8: Self energies to lowest order in the loop expansion. The external propagators should not be included.

in more detail by Carrington [20] and others [21]. In this improved perturbation theory the self energies in Fig. 8 are then to be calculated with the full propagators. In order for the Goldstone theorem to be satisfied, we know that  $\bar{D}_{ab}$  must have the form (83) involving the effective chemical potential  $\bar{\mu}$ . If this approximation method is consistent, the unknown function  $\bar{\mu}(T)$  can then be determined from the Pines-Hugenholtz relation (81) which now will constitute a gap equation.

There are other diagrams in addition to those given in Fig. 6 and Fig. 7, but they all vanish. Tadpole diagrams, such as , are zero because  $\langle \chi_1 \rangle = 0 = \langle \chi_2 \rangle$ . Diagrams with a  $\bar{D}_{12}$ -loop  are odd in the Matsubara frequency and thus vanish upon summation. The perturbative contributions to the diagonal self energies are non-zero. They depend in general on external momentum and energy. We have assumed a spatially constant condensate and can thus take the external momenta in the self energies to be zero. For the external energies we will also take these to be zero. This is obviously an approximation. At sufficiently high temperature, the zero-energy Matsubara modes should dominate the partition function and the approximation should then be reasonable. Especially for  $\Pi_{22}$  which we will need in the Pines-Hugenholtz relation (81), the approximation should be good because the pole in the corresponding real-time propagator is at zero energy. The off-diagonal self energies  $\Pi_{12}$  and  $\Pi_{21}$  in Fig. 8 both vanish at zero external energy. As earlier



stated, this follows to all orders in perturbation theory from time-reversal invariance.

From the diagrams in Fig. 8 we then find the self energy

$$\begin{aligned}\Pi_{11} = & \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[ 3(-\lambda) \bar{D}_{11}(k) + (-\lambda) \bar{D}_{22}(k) + 18(-\lambda v)^2 \bar{D}_{11}(k) \bar{D}_{11}(-k) \right. \\ & \left. + 2(-\lambda v)^2 \bar{D}_{22}(k) \bar{D}_{22}(-k) + 12(-\lambda v)^2 \bar{D}_{12}(k) \bar{D}_{12}(-k) \right] \quad (85)\end{aligned}$$

where we use  $k \equiv (\omega_n, \mathbf{k})$ . With the full propagators from (83) this can be simplified to

$$\Pi_{11} = -\frac{\lambda}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[ \frac{4\varepsilon + 2\bar{\mu}}{\omega^2 + \omega_n^2} - \frac{2\lambda v^2(10\varepsilon^2 + 4\varepsilon\bar{\mu} + 4\bar{\mu}^2 - 6\omega_n^2)}{(\omega^2 + \omega_n^2)^2} \right] \quad (86)$$

where  $\omega^2 = \varepsilon(\varepsilon + 2\bar{\mu})$ . The Matsubara summations are now easy to do. From the standard sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{\omega^2 + \omega_n^2} = \frac{\beta}{\omega} \left[ \frac{1}{2} + \frac{1}{e^{\beta\omega} - 1} \right] \quad (87)$$

follows directly by differentiation

$$\sum_{n=-\infty}^{\infty} \frac{\omega^2}{(\omega^2 + \omega_n^2)^2} = \frac{\beta}{2\omega} \left[ \frac{1}{2} + \frac{1}{e^{\beta\omega} - 1} + \beta\omega \frac{e^{\beta\omega}}{(e^{\beta\omega} - 1)^2} \right] \quad (88)$$

With  $\omega_n^2$  instead of  $\omega^2$  in the numerator on the left-hand side, we get the same result except for an opposite sign in front of the last term. The zero-temperature self energy is then found to be

$$\begin{aligned}\Pi_{11}(T=0) &= -\lambda \int \frac{d^3k}{(2\pi)^3} \left[ \frac{2\varepsilon + 4\bar{\mu}}{\omega} - \frac{\bar{\mu}(5\varepsilon^2 + 2\varepsilon\bar{\mu} + 2\bar{\mu}^2)}{\omega^3} \right] \\ &= -\lambda \int \frac{d^3k}{(2\pi)^3} \left[ \frac{2\varepsilon + \bar{\mu}}{\omega} - \frac{2\bar{\mu}(\varepsilon - \bar{\mu})^2}{\omega^3} \right] \quad (89)\end{aligned}$$

In the last line we have regrouped the terms to show the difference between  $\Pi_{11}$  and  $\Pi_{22}$  found below. The leading divergences will be cancelled by the counter-terms. This will explicitly be demonstrated in the case  $\Pi_{22}$  which we evaluate next. From the diagrams in Fig. 8 it is given as

$$\begin{aligned}\Pi_{22} &= \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[ 3(-\lambda) \bar{D}_{22}(k) + (-\lambda) \bar{D}_{11}(k) \right. \\ &\quad \left. + 4(-\lambda v)^2 \bar{D}_{11}(k) \bar{D}_{22}(-k) + 4(-\lambda v)^2 \bar{D}_{12}(k) \bar{D}_{12}(-k) \right] \\ &= -\frac{\lambda}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[ \frac{4\varepsilon + 6\bar{\mu}}{\omega^2 + \omega_n^2} - \frac{4\bar{\mu}(\varepsilon^2 + 2\varepsilon\bar{\mu} + \omega_n^2)}{(\omega^2 + \omega_n^2)^2} \right] \\ &= -\frac{\lambda}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{4\varepsilon + 2\bar{\mu}}{\omega^2 + \omega_n^2} \quad (90)\end{aligned}$$

A summation over Matsubara frequencies gives at non-zero temperature

$$\Pi_{22}(T) = -\lambda \int \frac{d^3k}{(2\pi)^3} \frac{4\varepsilon + 2\bar{\mu}}{\omega} \left[ \frac{1}{2} + \frac{1}{e^{\beta\omega} - 1} \right] \quad (91)$$

The first part of the integral which represents the zero-temperature self energy, is ultraviolet divergent. However, from (47) we see that we should add in the counter-terms  $\delta\mu - v^2\delta\lambda$ . Taking these from (49) and (50), with the replacement  $\mu \rightarrow \bar{\mu}$ , we obtain the renormalized and finite result

$$\Pi_{22}(T) = -\lambda \int \frac{d^3k}{(2\pi)^3} \left[ \frac{2\varepsilon + \bar{\mu}}{\omega} + \frac{\bar{\mu}}{\varepsilon} - 2 + \frac{4\varepsilon + 2\bar{\mu}}{\omega} \frac{1}{e^{\beta\omega} - 1} \right] \quad (92)$$

Similarly, an expansion for high momenta of the zero-temperature part of  $\Pi_{11}$  in (89) shows that the counter-terms  $\delta\mu - 3v^2\delta\lambda$  introduced in (47), again with  $\mu \rightarrow \bar{\mu}$ , exactly cancel the ultraviolet divergences in  $\Pi_{11}$ .

The assumption that the Goldstone theorem is fulfilled, i.e. the Pines-Hugenholtz relation (80), can now be shown to equivalent to the minimalization criterion (68) for a one-loop potential calculated with the full propagator  $\bar{D}_{ab}$ . Combining (55) with  $\mu \rightarrow \bar{\mu}$  in the one-loop term and (68) we find

$$\begin{aligned} v^2 &= \frac{\mu}{\lambda} - \int \frac{d^3k}{(2\pi)^3} \left[ \frac{2\varepsilon + \bar{\mu}}{\omega} + \frac{\bar{\mu}}{\varepsilon} - 2 + \frac{4\varepsilon + 2\bar{\mu}}{\omega} \frac{1}{e^{\beta\omega} - 1} \right] \\ &= \frac{\mu}{\lambda} + \frac{1}{\lambda} \Pi_{22}(T) \end{aligned} \quad (93)$$

when making use of the result (92). This is the same as equation (80) which embodies the Goldstone theorem .

As mentioned at the beginning of this section, the self energies are calculated using the dispersion relation  $\omega = \sqrt{\varepsilon(\varepsilon + 2\bar{\mu})}$ , i.e. neglecting terms of  $\mathcal{O}(\Pi_{11} - \Pi_{22})$ . Using the renormalized self energies, we see that the difference at zero temperature is

$$\Pi_{11} - \Pi_{22} = 2\lambda\mu \int \frac{d^3k}{(2\pi)^3} \left[ \frac{(\varepsilon - \mu)^2}{\omega^3} - \frac{1}{\varepsilon} \right] \quad (94)$$

The integral is finite at high momenta due to the counter-term, but is infrared divergent. This is caused by the exchange diagram in  $\Pi_{11}$  with two  $\bar{D}_{22}$ -lines which diverges as external energy and momenta are taken to zero. The infrared divergence signals the onset of long-distance effects which are not properly handled by the present one-loop approximation. As discussed by Kapusta [24], one can cure the divergence using a non-zero external energy. However, this is not a problem here.

At low temperature, the self energies are of  $\mathcal{O}(\lambda\mu^{3/2})$  and will be neglected all together, being small compared to  $\mu$ . On the other hand, at high temperature, i.e. near the critical temperature  $T_C$ ,  $\Pi_{22}$  is comparable to  $\mu$  and the self energies must be included. In this regime we can use the high-temperature limits of  $\Pi_{11}$  and  $\Pi_{22}$  which below are shown to be equal to leading order. Thus, in the temperature range where it should be considered,  $\Pi_{11} - \Pi_{22} = 0$ .

At high temperatures  $T \gg \bar{\mu}$  the exchange diagrams may be neglected. The dominating terms in the remaining diagrams go like  $\varepsilon/(\omega(e^{\beta\omega} - 1))$ , and the combinatorial prefactors are the same for the two self energies. Expanding (86) and (92) in powers of  $\bar{\mu}/T$  we find to leading order

$$\Pi_{22}(T) = \Pi_{11}(T) = -4\lambda\zeta(3/2) \left(\frac{mT}{2\pi}\right)^{3/2} \quad (95)$$

This result will be found if one uses free propagators  $D_{ab}$  in the self energy loops since  $\bar{\mu}$  can be neglected to leading order.

In the opposite limit  $T \ll \bar{\mu}$ , we can take  $\bar{\mu}$  approximately equal to  $\mu$ . Diagrams with a  $\bar{D}_{11}$ -loop are suppressed by a factor  $T^2/\mu^2$  compared to the others. We then obtain for the leading terms of the self energy

$$\Pi_{22}(T) = -\frac{10\lambda}{3\pi^2}(m\mu)^{3/2} - \frac{\lambda T^2 m^{3/2}}{6\mu^{1/2}} \quad (96)$$

The first part is seen to be of the order  $\mathcal{O}(\sqrt{\rho a^3})$  smaller than the chemical potential and the second term even smaller by a factor  $(T/\bar{\mu})^2$ . To leading order, we can thus neglect the self energies at these low temperatures and set  $\bar{\mu} = \mu$ . Thus, the effects of ring corrections on the thermodynamics of the system will only appear at high temperature.

### 4.3 Condensate and pressure at finite temperature

Including the effects of ring corrections, we can now write down the renormalized finite temperature effective potential

$$\begin{aligned} U_{eff}(v, \mu, T) &= -\frac{\mu}{2}v^2 + \frac{\lambda}{4}v^4 + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left( \omega - \varepsilon + \bar{\mu} - 2\lambda v^2 + \frac{\lambda^2 v^4}{2\varepsilon} \right) \\ &+ T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta\omega}) \end{aligned} \quad (97)$$

From (93) we know that it has a minimum at  $v = (\bar{\mu}/\lambda)^{1/2}$ . In the minimum we then have the pressure

$$\begin{aligned}
P(\mu, T) &= -U_{eff}(\mu, T) = \frac{1}{4\lambda}(\mu^2 - \Pi_{22}^2) - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \omega - \varepsilon - \bar{\mu} + \frac{\bar{\mu}^2}{2\varepsilon} \right] \\
&- T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta\omega})
\end{aligned} \tag{98}$$

The total density of particles is again found by using  $\rho = \partial P / \partial \mu$  which gives

$$\begin{aligned}
\rho &= \frac{\mu}{2\lambda} - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\varepsilon}{\omega} + \frac{\bar{\mu}}{\varepsilon} - 1 + \frac{2\varepsilon}{\omega(e^{\beta\omega} - 1)} \right] \\
&= \frac{\bar{\mu}}{2\lambda} + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\varepsilon + \bar{\mu}}{\omega} - 1 + \frac{2(\varepsilon + \bar{\mu})}{\omega(e^{\beta\omega} - 1)} \right]
\end{aligned} \tag{99}$$

while the density of particles in the condensate is

$$\rho_c = \frac{v^2}{2} = \frac{\bar{\mu}}{2\lambda} = \frac{\mu}{2\lambda} - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{2\varepsilon + \bar{\mu}}{\omega} + \frac{\bar{\mu}}{\varepsilon} - 2 + \frac{4\varepsilon + 2\bar{\mu}}{\omega(e^{\beta\omega} - 1)} \right] \tag{100}$$

The difference

$$\rho_e = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\varepsilon + \bar{\mu}}{\omega} \left( 1 + \frac{2}{e^{\beta\omega} - 1} \right) - 1 \right] \tag{101}$$

represents the density of particles in excited states with  $\mathbf{k} \neq 0$ . Except for the replacement of  $\mu$  with  $\bar{\mu}$ , this result agrees with what was obtained in [19]. Now both the hard-core repulsion and thermal fluctuations cause the excitation of particles from the condensate. With increasing temperatures more and more particles are in excited states and at a critical temperature where  $\bar{\mu} = 0$ , the condensate becomes zero. We then have a phase transition to the normal phase. In the following we will see that this critical temperature is the same as for a free Bose gas to the accuracy we are working.

Equation (99) gives upon inversion the chemical potential as a function of the particle density. The first integral on the right-hand side was done in Section 3.4, and we can now simply replace  $\mu$  with  $\bar{\mu}$  in Eq. (59). The second integral

$$I_T \equiv \int \frac{d^3k}{(2\pi)^3} \frac{\varepsilon}{\omega(e^{\beta\omega} - 1)} \tag{102}$$

must be done numerically. However, in the two important temperature ranges we can easily find good analytical approximations.

At low temperature we again take  $\bar{\mu} \simeq \mu$ . The dominant contributions to the integral comes from small values of  $\varepsilon$  and we can thus take  $\varepsilon + 2\mu \simeq 2\mu$  in the denominator. Then we have the result

$$I_T = \frac{\pi^2 m^{3/2} T^4}{60\mu^{5/2}} \quad (103)$$

Since the loop corrections are small, we can set  $\mu = 2\lambda\rho$  on the right hand side of (99). Inversion then trivially gives

$$\mu = 2\lambda\rho + \frac{8\lambda}{3\pi^2}(2m\lambda\rho)^{3/2} + \frac{\lambda\pi^2 m^{3/2} T^4}{30(2\lambda\rho)^{5/2}} \quad (104)$$

For comparison, the expression for the effective chemical potential takes the form

$$\bar{\mu} = 2\lambda\rho_c = 2\lambda\rho - \frac{2\lambda}{3\pi^2}(2m\lambda\rho)^{3/2} - \frac{\lambda m^{3/2} T^2}{6\sqrt{2\lambda\rho}} \quad (105)$$

Thus, at low  $T$ , the chemical potential *increases* with temperature, while the *effective* chemical potential, and thus the condensate density decreases with temperature. This temperature dependence is the same as obtained in the Bogoliubov approximation [25]. The pressure now follows from Eq. (98). As already explained, we can neglect the contribution coming from the self-energy. Thus

$$P = \frac{\mu^2}{4\lambda} - \frac{8m^{3/2}}{15\pi^2}\mu^{5/2} + P_T \quad (106)$$

where the temperature dependence is in the function

$$P_T = -T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta\omega}) = \frac{\pi^2 m^{3/2} T^4}{90\mu^{3/2}} \quad (107)$$

evaluated in the same approximation as above. The pressure at low temperatures is therefore

$$P(\rho) = \lambda\rho^2 + \frac{4m^{3/2}}{5\pi^2}(2\lambda\rho)^{5/2} + \frac{m^{3/2}\pi^2}{90} \frac{T^4}{(2\lambda\rho)^{3/2}} \quad (108)$$

At non-zero temperature we can obtain the energy density from the thermodynamic relation

$$\mathcal{E} = -\frac{\partial}{\partial\beta}(\beta P)_{\beta\mu} \quad (109)$$

which now gives

$$\mathcal{E} = \frac{2\pi a}{m}\rho^2 \left[ 1 + \frac{128}{15} \sqrt{\frac{a^3\rho}{\pi}} \right] + \frac{m^{3/2}}{240\pi} \frac{T^4}{\rho} \sqrt{\frac{\pi}{a^3\rho}} \quad (110)$$

It is seen that the effect of non-zero temperature comes in as a  $T^4$  contribution as in the Stefan-Boltzmann law for photons. In that case the excitations are mass-less because of gauge invariance, while here they are Goldstone bosons resulting from a broken, continuous symmetry.

In the opposite temperature range near the critical temperature  $T_C$ , we must include the effects of the self energies. They have previously been obtained in (95) which gives for the effective chemical potential (81)

$$\bar{\mu} = \mu - 4\lambda\zeta(3/2) \left( \frac{mT}{2\pi} \right)^{3/2} \quad (111)$$

This quantity also enters the total density of particles in (99) where we now to leading order can set  $\bar{\mu} = 0$ . The temperature-dependent integral (102) then simplifies to

$$I_T = \zeta(3/2) \left( \frac{mT}{2\pi} \right)^{3/2} \quad (112)$$

Since the chemical potential is now  $\mu = 2\lambda(\rho + I_T)$ , we have the relation

$$\mu = 4\lambda\rho - \bar{\mu} \quad (113)$$

for temperatures just below the  $T_C$ . The critical temperature is defined by  $\bar{\mu} = 0$ . Here the chemical potential takes the value  $\mu_C = 4\lambda\rho$ , and hence (111) gives

$$T_C = \frac{2\pi}{m} \left( \frac{\mu_C}{4\lambda\zeta(3/2)} \right)^{2/3} = \frac{2\pi}{m} \left( \frac{\rho}{\zeta(3/2)} \right)^{2/3} \quad (114)$$

which is just the textbook result [1]. As seen in Fig. 9, the thermodynamic chemical potential  $\mu$  increases continuously from zero to  $\mu_C$  at the critical temperature. On the other hand, the effective chemical potential  $\bar{\mu}$  decreases smoothly from zero temperature and is by definition zero at  $T_C$ .

In the general formula (98) we must now keep the contributions from the self energies when we calculate the pressure for temperatures just below the critical temperature. Thus

$$P = \frac{1}{4\lambda}(\mu^2 - \Pi_{22}^2) + \left( \frac{m}{2\pi} \right)^{3/2} T^{5/2} \zeta(5/2) - \left( \frac{mT}{2\pi} \right)^{3/2} \zeta(3/2)(\mu + \Pi_{22}) + \mathcal{O}(\bar{\mu}^2) \quad (115)$$

with  $\Pi_{22}$  given in (95). In order to compare with results in the literature, we write the pressure as a function of the particle density

$$P = \frac{2\pi}{m} a \rho^2 - \left( \frac{m}{2\pi} \right)^2 \zeta(3/2)^2 a T^3 + \left( \frac{m}{2\pi} \right)^{3/2} \zeta(5/2) T^{5/2} \quad (116)$$

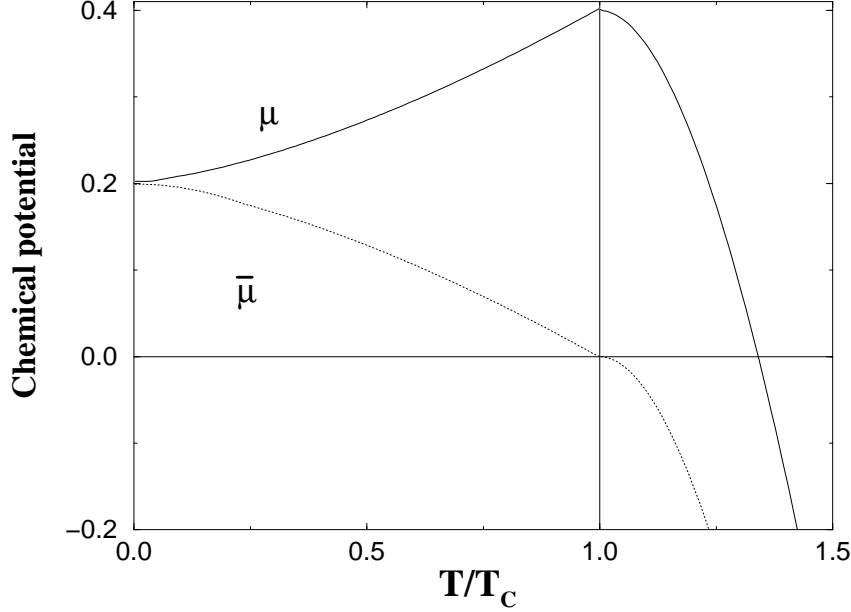


Figure 9: The thermodynamic and effective chemical potentials,  $\mu$  and  $\bar{\mu}$  plotted as functions of temperature for  $g = 0.1$  and  $m = 1$ . The results were obtained by matching the low and high temperature expressions for the chemical potentials.

Comparing with the result of Lee and Yang [8], we find that the second term has the wrong sign. The reason is the following: An expansion of the ring-corrected potential on powers of  $\lambda$  shows that the two-loop contribution is counted twice. Higher order diagrams are reproduced correctly. We must correct for this double counting by subtracting the two-loop contribution, as given in perturbation theory, from the obtained result. This contribution is given by the three uppermost diagrams in Fig. 6. Near  $T_C$  the propagators are equal to lowest order,  $D_{11} = D_{22} = \varepsilon/(\varepsilon^2 + \omega_n^2)$ . The total contribution is  $-2(m/2\pi)^2 \zeta(3/2)^2 a T^3$ . Subtracting this contribution, we thus have

$$P = \frac{2\pi}{m} a \rho^2 + \left(\frac{m}{2\pi}\right)^2 \zeta(3/2)^2 a T^3 + \left(\frac{m}{2\pi}\right)^{3/2} \zeta(5/2) T^{5/2} \quad (117)$$

which now agrees with [8]. This result also comes out in the mean field approximation [1, 31]. The equation for the chemical potential is not modified by this correction at the present order of accuracy. In the same approximation we also find the the energy density

$$\mathcal{E} = \frac{2\pi}{m} a \rho^2 - \sqrt{\frac{m}{2\pi}} \zeta(3/2)^2 a \rho T^{3/2} + 2\left(\frac{m}{2\pi}\right)^2 \zeta(3/2)^2 a T^3 + \frac{3}{2} \left(\frac{m}{2\pi}\right)^{3/2} \zeta(5/2) T^{5/2} \quad (118)$$

At the critical temperature these expressions reduce to

$$P(T_C) = \frac{2\pi}{m} \zeta(3/2)^{5/3} \zeta(5/2) \rho^{2/3} + \frac{4\pi}{m} a \rho^2 \quad (119)$$

and

$$\mathcal{E}(T_C) = \frac{3\pi}{m} \zeta(3/2)^{5/3} \zeta(5/2) \rho^{2/3} + \frac{4\pi}{m} a \rho^2 \quad (120)$$

which in the following are shown to equal the limits taken in the normal state, securing continuity at the critical temperature.

The special treatment at two-loop order could of course have been introduced at an earlier stage, but there would be no effects on the results found so far. When minimalizing the effective ring-corrected potential we discarded contributions from  $\partial\Pi/\partial v$ , being of  $\mathcal{O}(\lambda^2)$ . Since the contribution from differentiation of the two-loop potential is of the same order, this must be discarded as well, and the minimalization criterion remains unchanged. In the Appendix we briefly discuss how this is incorporated in the effective action for composite operators.

#### 4.4 Equation of state in the normal phase

At temperatures above  $T_C$ , the condensate density is zero and the effective potential has its only minimum at  $v = 0$ . The modified dispersion relation then reads  $\omega = \varepsilon - \bar{\mu}$  with effective chemical potential  $\bar{\mu} = \mu + \Pi_{22} < 0$ . Since  $v = 0$ , the exchange diagrams vanish and the expressions for the self energies are

$$\begin{aligned} \Pi_{11} = \Pi_{22} \equiv \Pi &= \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[ 3(-\lambda) \bar{D}_{22}(k) + (-\lambda) \bar{D}_{11}(k) \right] + \text{c.t.} \\ &= -\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\omega} - 1} \end{aligned} \quad (121)$$

The Bose-integral is standard and the self energy becomes

$$\Pi = -4\lambda \left( \frac{mT}{2\pi} \right)^{3/2} \text{Li}_{3/2}(e^{\beta\bar{\mu}}) \quad (122)$$

We have here introduced the polylogarithmic function

$$\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \quad (123)$$

with  $\text{Li}_n(1) = \zeta(n)$ . Near  $T_C$  we may approximate the exponential with unity and at  $T_C$  the result (95) is reproduced. To the present accuracy we may set  $\bar{\mu} = \mu$  in



the self energy. With vanishing condensate the classical and zero point terms do not contribute to the pressure, which simplifies to

$$P = \left(\frac{m}{2\pi}\right)^{3/2} T^{5/2} \text{Li}_{5/2}(\bar{z}) + 2\lambda \left(\frac{mT}{2\pi}\right)^3 \text{Li}_{3/2}^2(z) \quad (124)$$

with  $z = \exp(\beta\mu)$  and  $\bar{z} = \exp(\beta\bar{\mu})$ . To lowest order in  $\lambda$  the pressure can be written

$$P = \left(\frac{m}{2\pi}\right)^{3/2} T^{5/2} \text{Li}_{5/2}(z) - 2\lambda \left(\frac{mT}{2\pi}\right)^3 \text{Li}_{3/2}^2(z) \quad (125)$$

where the last term is the two-loop contribution. We have here used the following property of the polylogarithmic functions:

$$\frac{d}{dx} \text{Li}_n(x) = \frac{1}{x} \text{Li}_{n-1}(x) \quad (126)$$

Expanding in powers of  $\beta\mu$ , we find that this agrees with the result of Popov [23]. Again we want to write the pressure in terms of the particle density which now becomes

$$\rho = \left(\frac{mT}{2\pi}\right)^{3/2} \text{Li}_{3/2}(\bar{z}) + 4\lambda \left(\frac{m}{2\pi}\right)^3 T^2 \text{Li}_{1/2}(z) [\text{Li}_{3/2}(z) - \text{Li}_{3/2}(\bar{z})] \quad (127)$$

Inverting this relation, we find to lowest order in  $\lambda$

$$\mu = T \ln \text{Li}_{3/2}^{-1}(\rho \Lambda_T^3) + 4\lambda \rho \quad (128)$$

where  $\text{Li}_n^{-1}$  is the inverse of  $\text{Li}_n$  and  $\Lambda_T = \sqrt{2\pi/mT}$  is the thermal wave length. To the same order the self energy is  $\Pi = -4\lambda\rho$ . Thus, the effective chemical potential becomes

$$\bar{\mu} = \mu + \Pi = T \ln \text{Li}_{3/2}^{-1}(\rho \Lambda_T^3) \quad (129)$$

which equals the usual chemical potential for a free gas. Let us again refer to Fig. 9 where the chemical potentials  $\bar{\mu}$  and  $\mu$  are plotted as function of temperature. The difference  $\mu - \bar{\mu} = 4\lambda\rho$  found just below the critical temperature, remains constant in the normal phase where the potentials have the same temperature dependence.

As a function of density the pressure becomes

$$P = \frac{4\pi}{m} a \rho^2 + \rho T \left[ 1 - \frac{\Lambda_T^3 \rho}{2^{5/2}} \right] + \mathcal{O}(\rho^3) \quad (130)$$

The corresponding second virial coefficient reads

$$B_2 = -\frac{\Lambda_T^3}{2^{5/2}} \left[ 1 - 8a \sqrt{\frac{mT}{\pi}} \right] \quad (131)$$

From the relation  $\mathcal{E} = -\partial/\partial\beta(\beta P)|_{\beta\mu}$  we similarly have for the energy density

$$\mathcal{E} = \frac{4\pi}{m}a\rho^2 + \frac{3}{2}\rho T \left[ 1 - \frac{\Lambda_T^3 \rho}{2^{5/2}} \right] + \mathcal{O}(\rho^3) \quad (132)$$

again in agreement with earlier results [8, 1, 31]. At  $T_C$  these expressions coincide with the limits taken from sub-critical temperatures, found in the last subsection. Summing up, the Eqs. (108), (117) and (130) constitute the equation of state for the Bose gas at all temperatures.

## 5 Discussion and conclusion

The non-interacting Bose-Einstein gas in the condensed phase is not a thermodynamically stable system since its compressibility is infinite [1]. Also, the phase transition from the normal phase is special in that the correlation length diverges below the transition temperature. This unphysical behavior is directly related to the absence of a real spontaneous breakdown of the  $U(1)$  symmetry of the system with the accompanying Goldstone bosons which here are the phonon excitations.

With the introduction of a weak repulsion between the particles, the physics of the system is well-defined in both phases. We have here described the system using modern field-theoretic methods based on functional integrals. Instead of using the standard formalism with complex fields as used in the condensed matter literature [25], we find it more convenient to use real fields. The divergences in the loop integrals have been regulated by introducing a physical cutoff. Since these are proportional to powers of the cutoff, we could instead have used dimensional regularization where they consequently would be absent.

We have put special emphasis on enforcing the consequences of Goldstone's theorem. In the real-field formalism the important Pines-Hugenholtz theorem then takes a slightly different form. The thermodynamics is most directly obtained from the effective potential whose minimum gives the free energy of the system. At non-zero temperatures we find that the one-loop effective potential does not give a consistent description of the symmetry breakdown and that it must be improved by adding in ring corrections in a self-consistent way. It then becomes natural to introduce an effective chemical potential  $\bar{\mu}$  which acts as an order parameter, being positive below the critical temperature  $T_C$  and giving a non-zero condensate, and passing through zero at  $T_C$ . In this way we find  $\bar{\mu}$  is a more important variable in characterizing the phases of the system than the thermodynamic chemical potential  $\mu$ . In particular, for temperatures below  $T_C$  where the particle number concept loses some of its meaning, the physical significance of  $\mu$  is not clear.

Our results are derived to lowest order in the coupling between the particles. As such, most of the thermodynamic results are already in the literature dating back to the pioneering calculations of Lee, Yang and collaborators obtained by methods from statistical mechanics. What is new in addition to the systematic use of the effective chemical potential, is basically a more coherent derivation in a framework which has to a large extent been developed for corresponding relativistic systems in high energy physics. There is nothing preventing us in extending the calculations to higher orders in the interactions. One is then forced to consider the theory as an effective field theory which is non-renormalizable at very high energies, the coupling constant having dimensions of a length. As recently shown by Braaten and Nieto [32], the zero-temperature ground state energy to next order in the coupling constant can then be obtained in a most direct way using the renormalization group for this effective field theory.

With the recent introduction of magnetic traps, the experimental study of the thermodynamics of weakly interacting bosons have entered a new phase which will allow a much more detailed study of this important system. Here we have considered the particles in an open volume where the interactions dominate, while it is the confining potential which has the most important rôle in the trapped systems [4]. The number of particles in the system now is finite, but one can still use quantum field theory in the grand canonical ensemble [33]. Even with the large effort going on in the field at present, there is still a long way between experimental results and detailed verifications of the theoretical properties as derived for the open system here.

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## A Appendix

Here we give a brief review of the effective action for composite operators introduced by Cornwall, Jackiw and Tomboulis [29]. The method was originally presented in real time formalism, but we will use imaginary time, following [34]. A compact notation with  $\int_x \equiv \int d^d x$  and  $J_x \equiv J(x)$  is used. One starts by defining the generating functional

$$\begin{aligned} Z[J, K] &= \exp(-W[J, K]) \\ &= \int D\phi \exp \left[ - \left( S[\phi] + \int_x \phi_x J_x + \frac{1}{2} \int_{xy} \phi_x K_{xy} \phi_y \right) \right] \end{aligned} \quad (133)$$

where the field(s)  $\phi$  is coupled both to a local source  $J_x$  and a bilocal source  $K_{xy}$ .  $S[\phi]$  is the classical Euclidean action. The background field  $\Phi$  and the propagator  $G$  are defined by

$$\frac{\delta W[J, K]}{\delta J_x} = \Phi_x \quad (134)$$

$$\frac{\delta W[J, K]}{\delta K_{xy}} = \frac{1}{2}[\Phi_x \Phi_y + G_{xy}] \quad (135)$$

A double Legendre transform then produces the effective action

$$\Gamma[\Phi, G] = W[J, K] - \int_x \Phi_x J_x - \frac{1}{2} \int_{xy} [\Phi_x K_{xy} \Phi_y + G_{xy} K_{yx}] \quad (136)$$

where  $J$  and  $K$  are eliminated using (134) and (135). The sources are then related to  $\Gamma[\Phi, G]$  via the equations

$$\frac{\delta \Gamma[\Phi, G]}{\delta \Phi_x} = -J_x - \int_y K_{xy} \Phi_y \quad (137)$$

$$\frac{\delta \Gamma[\Phi, G]}{\delta G_{xy}} = -\frac{1}{2} K_{xy} \quad (138)$$

In order to establish a series expansion for the effective action one must shift the field in the classical action  $S[\phi] \rightarrow S[\Phi + \phi]$ . From this the free propagator in background field

$$D_{xy}^{-1}(\Phi) = \frac{\delta^2 S[\Phi + \phi]}{\delta \phi_x \delta \phi_y} \Big|_{\phi=0} \quad (139)$$

is obtained. A careful rearrangement of terms leads to the series [29]

$$\Gamma[\Phi, G] = S[\Phi] + \frac{1}{2} \text{Tr} \ln D_0 G^{-1} + \frac{1}{2} \text{Tr}[D^{-1} G - 1] + \Gamma_2[\Phi, G] \quad (140)$$

$\Gamma_2[\Phi, G]$  is the sum of all two-particle-irreducible graphs obtained from the interaction terms in  $S[\Phi + \phi]$  with propagator  $G$  and  $D_0 = D(\Phi = 0)$ . The conventional effective action is now obtained from the equations

$$\Gamma[\Phi] = \Gamma[\Phi, G_0] \quad (141)$$

$$\frac{\delta \Gamma[\Phi, G]}{\delta G} \Big|_{G=G_0} = 0 \quad (142)$$

As seen from (138) this is equivalent to letting  $K = 0$ .

We now want to compare the effective action (140) to the results obtained in the previous sections.  $\Phi$  is then the background field  $(v, 0)$  and  $\phi = (\chi_1, \chi_2)$ . We only need  $\Gamma$  to  $\mathcal{O}(\lambda)$ . Using  $G^{-1} = D^{-1} - \Pi$  we find for the various terms

$$\frac{1}{2}\text{Tr}[D^{-1}G - 1] = \frac{1}{2}\text{Tr}D\Pi + \mathcal{O}(\lambda^2) \quad (143)$$

$$\frac{1}{2}\text{Tr}\ln G^{-1} = \frac{1}{2}\text{Tr}[\ln D^{-1} - D\Pi + \dots] \quad (144)$$

Hence, to this order, the effect of the third term in (140) is to cancel the  $\mathcal{O}(\lambda)$  contribution, in terms of the propagator  $D$  in the one-loop term. To the same order  $\Gamma_2$  is simply the sum of two-loop graphs with propagator  $G$ . Omitting a constant due to  $\text{Tr}\ln D_0$  we may to this order write

$$\Gamma[v, \Pi] = S[v] + \frac{1}{2}\text{Tr}\ln G^{-1} - \frac{1}{2}\text{Tr}D\Pi + \text{two-loop} \quad (145)$$

If we now make the ansatz for  $G = [D^{-1} - \Pi]^{-1}$ , as we did assuming the Goldstone theorem, this is seen to equal our result for the ring-corrected effective potential.

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